

α) Είναι:

$$\begin{aligned}\Delta &= [-(\alpha^2 + \beta^2)]^2 - 4 \cdot \alpha\beta \cdot \alpha\beta = \\ &= \alpha^4 + 2\alpha^2\beta^2 + \beta^4 - 4\alpha^2\beta^2 = \\ &= \alpha^4 - 2\alpha^2\beta^2 + \beta^4 = (\alpha^2 - \beta^2)^2\end{aligned}$$

β) Η εξίσωση έχει δύο πραγματικές και άνισες ρίζες αν και μόνο αν:

$$\begin{aligned}\Delta > 0 &\Leftrightarrow (\alpha^2 - \beta^2)^2 > 0 \Leftrightarrow \\ \Leftrightarrow (\alpha^2 - \beta^2)^2 \neq 0 &\Leftrightarrow \alpha^2 - \beta^2 \neq 0 \Leftrightarrow \\ \Leftrightarrow (\alpha - \beta)(\alpha + \beta) \neq 0 &\Leftrightarrow \\ \Leftrightarrow (\alpha - \beta \neq 0 \text{ και } \alpha + \beta \neq 0) &\Leftrightarrow \\ \Leftrightarrow (\alpha \neq \beta \text{ και } \alpha \neq -\beta)\end{aligned}$$

Οι ρίζες της εξίσωσης είναι οι:

$$\begin{aligned}x_{1,2} &= \frac{-\beta \pm \sqrt{\Delta}}{2\alpha} = \frac{-[-(\alpha^2 + \beta^2)] \pm \sqrt{(\alpha^2 - \beta^2)^2}}{2\alpha\beta} = \frac{\alpha^2 + \beta^2 \pm (\alpha^2 - \beta^2)}{2\alpha\beta} = \\ &= \begin{cases} \frac{\alpha^2 + \beta^2 + \alpha^2 - \beta^2}{2\alpha\beta} = \frac{2\alpha^2}{2\alpha\beta} = \frac{\alpha}{\beta} \\ \frac{\alpha^2 + \beta^2 - \alpha^2 + \beta^2}{2\alpha\beta} = \frac{2\beta^2}{2\alpha\beta} = \frac{\beta}{\alpha} \end{cases}\end{aligned}$$

γ) Ισχύει ότι:

$$\begin{aligned}(1 + x_1)(1 + x_2) &\geq 4 \Leftrightarrow \left(1 + \frac{\alpha}{\beta}\right) \left(1 + \frac{\beta}{\alpha}\right) \geq 4 \Leftrightarrow \\ \Leftrightarrow 1 + \frac{\beta}{\alpha} + \frac{\alpha}{\beta} + \frac{\beta}{\alpha} \cdot \frac{\alpha}{\beta} &\geq 4 \Leftrightarrow \\ \Leftrightarrow 1 + \frac{\beta}{\alpha} + \frac{\alpha}{\beta} + 1 &\geq 4 \Leftrightarrow \\ \Leftrightarrow \frac{\beta}{\alpha} + \frac{\alpha}{\beta} &\geq 2 \iff \\ \Leftrightarrow \alpha\beta \frac{\beta}{\alpha} + \alpha\beta \frac{\alpha}{\beta} &\geq 2\alpha\beta \Leftrightarrow \\ \Leftrightarrow \beta^2 + \alpha^2 - 2\alpha\beta &\geq 0 \Leftrightarrow (\alpha - \beta)^2 \geq 0,\end{aligned}$$

το οποίο ισχύει για κάθε θετικούς αριθμούς α, β .